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Restriction of divisor classes to hypersurfaces in characteristic p

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Abstract

The injectivity of the restriction homomorphism on divisor class groups to hypersurfaces has been studied by Grothendieck, Danilov, Lipman, and Griffith & Weston, among others. In particular, when A is a Noetherian normal domain of equicharacteristic zero and A/fA satisfies R_1, Spiroff established a map $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$, where $(A/fA)'$ represents the integral closure of A/fA, and gave some conditions for injectivity. In this paper, the authors continue in the same vein, but in the case of characteristic $p > 0$. In addition, when the hypersurface A/fA is normal, they provide further enlightenment about the kernel of $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$. Finally, using the second author's previous results, they exhibit a new class of examples for which the kernel is non-trivial.

1. Introduction

In the second author's article [21], the following questions are considered: If $(A, \mathfrak{m})$ is a local normal domain, $f_1, f_2, \ldots$ is a sequence of elements in A such that $\lim_{n\to\infty} f_n = 0$, and each A/f_nA satisfies the Serre regularity condition R_1, must it be the case that no non-trivial divisor class can lie in all of the kernels of $\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n)$, where A'_n denotes the integral closure of $A_n := A/f_nA$? That is, must every non-trivial divisor class have non-trivial image under at least one of the maps $\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n)$? Secondly, if the answer is "yes," are there good conditions so that the intersection $\bigcap_{n=1}^{\infty} \mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n))$, becomes zero at some predictable finite stage?

The first question was inspired by an article of C. Miller [17], in which a similar problem concerning power series rings $A[\![T]\!]$ was considered. In [21, Theorem 3.1], Spiroff obtains

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a general affirmative answer to the first question. In addition, a partial positive answer to the second question is obtained as well. In particular, Spiroff shows that for a local isolated singularity of dimension greater than or equal to four and of equal characteristic zero, an affirmative answer to the second question can be established, provided the ring in question possesses a non-trivial small Cohen–Macaulay module [21, Theorem 4.1]. The argument relies on a blend of ideas taken from standard commutative algebra, Hochschild cohomology, and lifting properties of small Cohen–Macaulay modules in the style of Yoshino [23, Section 6] and Popescu [18, Section 1].

The basic philosophy being espoused here is that the deeper f lies in powers of the maximal ideal, the better the injective behavior of the group homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$. Put another way, the behavior of the divisor class group on any collection of hypersurfaces, as described above, should reflect most elementary properties of the divisor class group of A (e.g., finite generation, torsion, etc.).

In Section 2, we provide some basic facts concerning divisor classes and the maps between divisor class groups. In Section 3, we consider graded k-algebras, where k is a perfect field of characteristic $p > 0$. We study the restriction homomorphism $\mathrm{Cl}(S) \to \mathrm{Cl}((S/fS)')$, where $S_0 = k$, S is a normal domain, and f is a homogeneous element, in a sufficiently high power of the irrelevant maximal ideal of S, such that S/fS satisfies R_1. In addition, we require $\dim S$ to be greater than or equal to four and $\mathrm{Spec}(S) - \mathfrak{m}$ to be locally regular, where $\mathfrak{m} = S_+$. In case $S = S_0[S_1]$, note that our requirements may be expressed by stating that $V = \mathrm{Proj}(S)$ is smooth over k with dimension greater than or equal to three. Within this context, the hypersurface W defined by $f = 0$ must be smooth in codimension less than or equal to one. The induced homomorphism $S(V) \to S(W)'$ provides an equivalent way of computing the kernel of $j^*: \mathrm{Cl}(V) \to \mathrm{Cl}(W)$, where $j: W \to V$ represents inclusion. That is, $\mathrm{Ker}(j^*)$ is naturally equivalent to the kernel of the homomorphism $\mathrm{Cl}(S(V)) \to \mathrm{Cl}(S(W)')$. (See Section 3 for further discussion.)

It is well known that the graded rings S described above, because they are isolated singularities, have small Cohen–Macaulay modules (see M. Hochster [13, Corollary 5.12] for a discussion of this fact). We use this fact to establish that the kernel of $\mathrm{Cl}(S) \to \mathrm{Cl}((S/fS)')$ is at worst a bounded p-group. Our proof makes use of properties encountered through lifting small Cohen–Macaulay modules as developed by Yoshino [23, Section 6] and Popescu [18, Section 1] in the context of Cohen–Macaulay rings. In Appendix A, we supply the slightly more general version (of their work) that we need.

Section 4 considers the case where all the hypersurfaces A/fA under consideration are normal, rather than simply R_1. We show that when $[\mathfrak{a}]$ is a non-trivial element in the kernel of $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$, then $f \cdot \mathrm{Ext}^1_A(\mathfrak{a}, -)$ is not identically zero. Using this fact, we establish the main result of this section: that, independent of characteristic, if A is an isolated singularity of dimension greater than three that possesses a small Cohen–Macaulay module M and f lies in a sufficiently high power of $\mathfrak{m}$, then $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$ is injective. We conclude the section by providing a connection of these results, in the case of characteristic $p > 0$, with those of Section 3.

Finally, in Section 5, we combine Spiroff's theorem [21, Theorem 3.1] with results of Danilov [6, Section 5] in order to demonstrate that the homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ may have non-trivial kernels when f is not required to lie sufficiently deep inside of the maximal ideal. This should come as no surprise in view of Danilov's results in [5–7] for the restriction homomorphism $\mathrm{Cl}(A[\![T]\!]) \to \mathrm{Cl}(A)$. However, in Theorem 5.1, our ambient ring represents an isolated singularity of dimension greater than or equal to three. This requirement means that our ambient ring can not be of the form $A[\![T]\!]$. Nor can it be a complete intersection when its dimension exceeds three since the presence of a non-zero divisor class would be in violation of Grothendieck's results [11, Chapter XI], that such a ring is a unique factorization domain. Thus, our example for which the kernel of the restriction of divisor classes is non-trivial is exclusive of the two most quoted types of examples.

2. Preliminaries

All rings are assumed to be Noetherian. For most of our deliberations, the notion of divisor class group of a normal domain A will follow the account described by J. Lipman [15, Section 0] and recounted in later articles [10, Section 1] and [21, Section 2]. By definition, the divisor class group of A, denoted by $\mathrm{Cl}(A)$, is the group of isomorphism classes of reflexive ideals of A. (Equivalently, consider the rank one reflexive A-modules.) To be specific, the *class* of an ideal $\mathfrak{a}$ has the property $[\mathfrak{a}] = [\mathfrak{a}^{**}]$, where $(-)^* = \mathrm{Hom}_A(-, A)$. So each ideal class contains a unique reflexive representative, up to isomorphism. Multiplication is defined by $[\mathfrak{a}] \cdot [\mathfrak{b}] = [(\mathfrak{a} \otimes \mathfrak{b})^{**}]$. The class of any principal ideal represents the identity class.

Let $f = 0$ be an irreducible hypersurface in $\operatorname{Spec} A$ such that A/fA is regular in codimension less than or equal to one (i.e., A/fA satisfies the Serre regularity condition R_1). Denote by $(-)'$ the integral closure of A/fA. We collect a few observations about calculations in $\mathrm{Cl}(A)$ and $\mathrm{Cl}((A/fA)')$ that will facilitate our arguments in Sections 3–5.

(2.1) $[\mathfrak{a}] = [A]$ if and only if $\mathfrak{a} \cong I$, where the ideal I contains a regular 2-sequence of A.

(2.2) There is a homomorphism of divisor class groups $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$. This homomorphism takes the class of the reflexive ideal $\mathfrak{a}$ to $[(\mathfrak{a}/f\mathfrak{a})^{**}] \in \mathrm{Cl}((A/fA)')$, where duals here are taken with respect to $(A/fA)'$.

(2.3) If the divisor class of a reflexive ideal $\mathfrak{a} \subset A$ is in the kernel of the homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$, then $\mathrm{Hom}_A(\mathfrak{a}, N) \cong N$ for any finitely-generated reflexive $(A/fA)'$-module N. More specifically, with $B = (A/fA)'$:

$$\mathrm{Hom}_A(\mathfrak{a}, N) \cong \mathrm{Hom}_B\big((\mathfrak{a} \otimes_A B)^{**}, N\big) \cong N,$$

where the first isomorphism involves methods found in [2, Section 4], and the second uses the fact that $[\mathfrak{a}]$ is in the kernel of $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$. (See [21, Theorem 4.1].) Included in the above statement is the case $N = (A/fA)'$. The converse of the statement is true as well.

(2.4) If M is a finitely-generated maximal Cohen–Macaulay A-module (hereafter referred to as a *small Cohen–Macaulay module*), then the A/fA-module $\overline{M} = M/fM$ is also an $(A/fA)'$-module. Therefore, if $[\mathfrak{a}] \in \mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)'))$, then $\mathrm{Hom}_A(\mathfrak{a}, \overline{M}) \cong \overline{M}$.

(2.5) There are a few occasions that we will want to appeal to the Bourbaki description of divisor class group [3, Chapter 7] from the additive point of view—the notion of *attached divisor classes*. It has two advantages, the first of which is that an attached divisor class is defined for any finitely-generated module M. More specifically, there is a free submodule F of M such that M/F is torsion. The *divisor attached to M* is $\chi(M/F)$, where:

$$\chi(M/F) = \sum_{\mathrm{ht}\,\mathfrak{p}=1} l(M/F)_{\mathfrak{p}} \cdot \mathfrak{p}.$$

Define $[M]$ to be $[\chi(M/F)]$. The second advantage is that classes of attached divisors are additive on short exact sequences. To be specific, if $0 \to N \to M \to L \to 0$ is a short exact sequence of finitely-generated modules, then $[M] = [N] + [L]$. For a fractional ideal $\mathfrak{a}$, the attached divisor of $\mathfrak{a}$ is the same as the class of $\mathfrak{a}$ in the multiplicative sense. As long as there is no confusion in a given context, we will simply denote a *class* by $[M]$, or $[\mathfrak{a}]$, when using either structure.

In Section 4, we work in the context of $\mathbb{N}$-graded rings S over a field k. (Our fields will be required to be perfect.) This simply means that S is a finitely-generated graded k-algebra, $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$, with $S_0 = k$, and where $\mathfrak{m} = S_+$ denotes the graded maximal ideal. When S is a normal domain, there is a natural isomorphism $\mathrm{Cl}(S) \to \mathrm{Cl}(S_{\mathfrak{m}})$, by [20, Proposition 6]; so $\mathrm{Cl}(S) \hookrightarrow \mathrm{Cl}(\widehat{S})$, where $\widehat{S}$ is the completion of $S_{\mathfrak{m}}$ at the graded maximal ideal. For any prime element $f \in S$ such that S/fS satisfies R_1, there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Cl}(S) & \hookrightarrow & \mathrm{Cl}(\widehat{S}) \\ \downarrow & & \downarrow \\ \mathrm{Cl}\big((S/fS)'\big) & \hookrightarrow & \mathrm{Cl}\big((\widehat{S/fS})'\big). \end{array}$$

Therefore, the kernel of the homomorphism $\mathrm{Cl}(S) \to \mathrm{Cl}((S/fS)')$ lies in the kernel of $\mathrm{Cl}(\widehat{S}) \to \mathrm{Cl}((\widehat{S/fS})')$, and questions concerning injectivity of the first map can be transferred to the second. Thus, we can complete S.

By Cohen Structure Theory, there is a regular local ring R such that $R \hookrightarrow S$ is a module-finite extension. Let $\Lambda = S \otimes_R S$ be the enveloping algebra, $\mu : \Lambda \to S$ the multiplication map, $\mathfrak{J}$ the kernel of μ, and $\eta = \mathrm{Ann}_\Lambda \mathfrak{J}$. Then the *Noetherian different* of S with respect to R is $\mu(\eta)$ and is denoted by $\mathfrak{N}_{S/R}$. A convenient reference for this material is M. Auslander and D. Buchsbaum [1]. The *ideal of Noetherian differents* $\mathfrak{N}_S$, composed of all the $\mathfrak{N}_{S/R}$, defines the singular locus of S; i.e., $\mathfrak{p}$ contains $\mathfrak{N}_S$ if and only if $S_{\mathfrak{p}}$ is not regular. (See [23, (4.2)] for more details on the matter.)

For unexplained terminology in commutative algebra, we suggest Matsumura's book [16] as a reference, and likewise for references to algebraic geometry, we suggest Hartshorne's book [12].

3. Characteristic $p > 0$

Let k be a perfect field of positive characteristic p and let S denote an $\mathbb{N}$-graded ring of dimension greater than or equal to four such that S is normal and $S_0 = k$. In addition, we assume that $\mathrm{Spec}(S) - \mathfrak{m}$ is regular, where $\mathfrak{m} = S_+$ is the graded maximal ideal. In the geometric setting where $S = S_0[S_1]$, we can achieve the regularity condition by requiring $V = \mathrm{Proj}(S)$ to be smooth over k.

Next, let f be a homogeneous prime element in $\mathfrak{m}$ such that the factor ring S/fS is regular in codimension less than or equal to one. We concern ourselves with the injective behavior of the induced homomorphism "restriction" of divisor class groups $\mathrm{Cl}(S) \to \mathrm{Cl}((S/fS)')$. See Section 2 for more details of this construction.

Following the lead in [21, Theorem 4.1], we wish to show the restriction homomorphism is injective, or nearly injective, when f is suitably "deep" in $\mathfrak{m}$. In particular, we study the situation where $x_1, x_2, \ldots, x_d$ is a system of parameters for S that is contained in the *ideal of Noetherian differents* $\mathfrak{N}_{\widehat{S}}$, where $\widehat{S}$ is the $\mathfrak{m}$-adic completion of S. (See Section 2.) Our basic requirement on f is that it lie in the parameter ideal $(x_1^2, x_2^2, \ldots, x_d^2)$. In Theorem 3.5, we argue that $\mathrm{Ker}(\mathrm{Cl}(S) \to \mathrm{Cl}((S/f)')$ is at worst a bounded p-group, in this case. Although our line of argument in the proofs of Theorem 3.5 follows a similar pattern as the second author's article [21, Section 4] for the case of equicharacteristic zero, the ingredients that go into the proof of [21, Theorem 4.1] need to be refined and made suitable for application in positive characteristic. We highlight a few of the necessary changes in the lemmas and observations that precede the proof of Theorem 3.5.

Remark 3.1. The graded ring S has a nonzero small Cohen–Macaulay module M. In our setting of characteristic $p > 0$, this fact was first noticed by Hartshorne–Peskine–Szpiro [19] in dimension three. An account is given by Hochster [13, Corollary 5.12] that covers the case at hand.

Lemma 3.2. *Let S be as above and let M be a small Cohen–Macaulay module. Then there is a system of parameters, $x_1, x_2, \ldots, x_d$, that depends only on $\widehat{S}$, such that* $(x_1, x_2, \ldots, x_d)\,\mathrm{Ext}^1_{\widehat{S}}(\widehat{M}, -) \equiv 0$.

Proof. The argument given in [21, Claims 4.3 and 4.4] applies to $\widehat{S}$ here as well. Namely, the ideal $\mathfrak{N}_{\widehat{S}}$ is $\widehat{\mathfrak{m}}$-primary and has the property that $\mathfrak{N}_{\widehat{S}}\,\mathrm{Ext}^1_{\widehat{S}}(\widehat{M}, -) \equiv 0$. Therefore, we may take a graded system of parameters $x_1, x_2, \ldots, x_d$ in $\mathfrak{N}_{\widehat{S}}$ such that $(x_1, x_2, \ldots, x_d)\,\mathrm{Ext}^1_{\widehat{S}}(\widehat{M}, -) \equiv 0$. □

Notation 3.3. For a system of parameters $x_1, \ldots, x_d$, set $\mathbf{x}^{(2)} = (x_1^2, \ldots, x_d^2)$.

Remark 3.4 ([23, Remark 6.19], [18, Section 1]). Let the graded system of parameters $x_1, x_2, \ldots, x_d$ in S be as in (3.2). If f is a homogeneous prime element in $\mathbf{x}^{(2)}$ and if $\overline{M} = M/fM$ is a small Cohen–Macaulay module over S/fS, then it has a unique lifting to S should it have any lifting at all. Since the Yoshino and Popescu arguments assume the ring is also Cohen–Macaulay, we provide a brief exposition of this result in Appendix A.

Theorem 3.5. *Let k be a perfect field of positive characteristic p and let S denote an $\mathbb{N}$-graded ring of dimension greater than or equal to four such that $S_0 = k$ and S is a normal domain. We assume that* $\operatorname{Spec}(S) - \mathfrak{m}$ *is regular, where $\mathfrak{m} = S_+$. Let $x_1, \ldots, x_d$ be a system of parameters contained in $S \cap \mathfrak{N}_{\widehat{S}}$ and suppose that f is a homogeneous prime element in $\mathbf{x}^{(2)}$ such that S/fS satisfies R_1. Then the kernel K of* $\operatorname{Cl}(S) \to \operatorname{Cl}((S/fS)')$ *is at worst a bounded p-group. Moreover, if* $\gcd(\operatorname{rank} M, p) = 1$, *where M is the small Cohen–Macaulay module of* (3.1), *then* $\operatorname{Cl}(S) \to \operatorname{Cl}((S/fS)')$ *is injective.*

Proof. As discussed in Section 2, we can complete S. Thus, assume S is complete, and construct the ideal $\mathfrak{N}_S$. Choose a system of parameters $x_1, x_2, \ldots, x_d$ in $\mathfrak{N}_S$ and let $f \in \mathbf{x}^{(2)}$ be a prime element such that S/fS satisfies R_1. Set $B = (S/fS)'$. Suppose $\mathfrak{a}$ is a reflexive ideal of S whose divisor class lies in the kernel of $\operatorname{Cl}(S) \to \operatorname{Cl}(B)$. Then $[(\mathfrak{a} \otimes_S B)^*] = [B]$, where the dual is taken with respect to B.

Let M be a graded small Cohen–Macaulay S-module, as per (3.1). Then we have the short exact sequence $0 \to M \xrightarrow{\cdot f} M \to \overline{M} \to 0$. The same argument used in [21, proof of Theorem 4.1] (which requires that $\dim S \geqslant 4$) shows the exactness of

$$0 \longrightarrow \operatorname{Hom}_S(\mathfrak{a}, M) \xrightarrow{\cdot f} \operatorname{Hom}_S(\mathfrak{a}, M) \longrightarrow \operatorname{Hom}_S(\mathfrak{a}, \overline{M}) \longrightarrow 0. \qquad (\dagger)$$

Moreover, $\operatorname{Hom}_S(\mathfrak{a}, \overline{M}) \cong \overline{M}$, as per (2.4). Thus, by ($\dagger$), $\operatorname{Hom}_S(\mathfrak{a}, M)$ is a lifting of $\overline{M}$. But then (3.4) implies that $\operatorname{Hom}_S(\mathfrak{a}, M) \cong M$. Let $r = \operatorname{rank}(M)$. As observed in [21, proof of Theorem 4.1] and also in [17, Lemma 6.3]:

$$[\operatorname{Hom}_S(\mathfrak{a}, M)] = -r[\mathfrak{a}] + [M].$$

Thus, $r[\mathfrak{a}] = 0$ in $\operatorname{Cl}(S)$, using the additive notation of (2.5). Now by [10, Theorem 1.2], K contains no elements of order prime to p. Thus, $|[\mathfrak{a}]|$ is some power of p, which means that $r = p^e l$, where $e \geqslant 1$ and $\gcd(p, l) = 1$. Hence, K is a bounded p-group. Moreover, if $\gcd(r, p) = 1$, then $\operatorname{Cl}(S) \to \operatorname{Cl}(B)$ is injective. □

Remark 3.6. Note that the above proof gives us a description of the kernel. To be specific, we have shown that:

$$K \subset \{[\mathfrak{a}] \in \operatorname{Cl}(S) \mid \operatorname{Hom}_S(\mathfrak{a}, M) \cong M \text{ for any small Cohen–Macaulay } M\}.$$

We end this section with some remarks about the geometric interpretation of our result. Here we assume that $S = S_0[S_1]$, $S_0 = k$, and $\operatorname{Proj}(S) = V$ is smooth over k. Then the homogeneous element f defines a hypersurface H in V which is smooth in codimension less than or equal to one. Within this framework there is a commutative diagram in which the column homomorphisms amount to restriction (see Hartshorne [12, Section II.6] and Samuel [20, p. 159]):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Cl}(V) & \longrightarrow & \mathrm{Cl}(S) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Cl}(H) & \longrightarrow & \mathrm{Cl}((S/fS)') & \longrightarrow & 0. \end{array}$$

It follows that there is a natural identification between the kernels of the maps $\mathrm{Cl}(V) \to \mathrm{Cl}(H)$ and $\mathrm{Cl}(S) \to \mathrm{Cl}((S/fS)')$. Thus, the results on the divisor class groups of Theorem 3.5 apply to $\mathrm{Cl}(V) \to \mathrm{Cl}(H)$ as well.

4. Normal hypersurfaces

In this section, rather than requiring our hypersurfaces to only satisfy R_1, as in Section 3, we consider the case where the hypersurfaces are all normal. Consequently, we obtain more information about the kernels of the restriction maps than in (3.5). Moreover, our results are independent of characteristic. However, we will provide a connection between these new results and those of the previous section. Our main observation is the following.

Proposition 4.1. *Suppose that A and A/fA are normal local domains and that $[\mathfrak{a}]$ is a non-trivial element in the kernel of $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$. Then $f \cdot \mathrm{Ext}^1_A(\mathfrak{a}, -)$ is not identically zero.*

Proof. Form a short exact sequence $0 \to K \to F \to \mathfrak{a} \to 0$, where F is A-free. Suppose $f \cdot \mathrm{Ext}^1_A(\mathfrak{a}, K) = 0$. For the homomorphisms $\cdot f$ and $F \to \mathfrak{a}$ there is a pullback diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & K \oplus \mathfrak{a} & \longrightarrow & \mathfrak{a} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \cdot f} & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & \mathfrak{a} & \longrightarrow & 0. \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bar{\mathfrak{a}} & = & \bar{\mathfrak{a}} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

(Note that the top row is split exact since it is obtained by multiplying the bottom row by f.) Dualizing the middle column with respect to A gives:

$$F^* \hookrightarrow K^* \oplus \mathfrak{a}^* \longrightarrow \mathrm{Ext}^1_A(\bar{\mathfrak{a}}, A) \longrightarrow 0.$$

Note that $\mathrm{Ext}^1_A(\bar{\mathfrak{a}}, A) \cong \mathrm{Hom}_A(\bar{\mathfrak{a}}, \overline{A}) \cong \overline{A}$. Hence $\mathrm{pd}_A(K^* \oplus \mathfrak{a}^*) \leqslant 1$, which implies that $\mathrm{pd}_A \mathfrak{a}^* \leqslant 1$. In other words, $\mathfrak{a}^*$ has an FFR. Thus, $\mathfrak{a}^* \cong A$ [3, p. 533]; hence, $\mathfrak{a} \cong A$. Contradiction. □

As we stated in the introduction, this result will be instrumental in obtaining further information about the injectivity of the maps $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$. However, before arriving at any important conclusions or connections to the previous sections, we need some preliminary observations.

Lemma 4.2. *Let $(A, \mathfrak{m})$ be a local domain of dimension greater than or equal to two and let M be a torsion-free finitely-generated A-module that is locally free on* $\mathrm{Spec}(A) - \mathfrak{m}$. *Then there is a system of parameters $x_1, \ldots, x_d$ for A such that M_{x_i} is A_{x_i}-free for $i = 1, \ldots, d$.*

Proof. Let the ideal I be generated by all $y \neq 0$ such that M_y is A_y-free. If I is not $\mathfrak{m}$-primary, then choose $\mathfrak{p} \in \mathrm{Spec}(A) - \mathfrak{m}$ containing I. Since M is locally free on the punctured spectrum of A, $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. Choose a maximal linearly independent set $\mathfrak{L}$ in M such that $F = \bigoplus_{\lambda \in \mathfrak{L}} A\lambda$ has the property $F_\mathfrak{p} = M_\mathfrak{p}$. Then $0 \to F \to M \to W \to 0$, where W is torsion, and $F_\mathfrak{p} \cong M_\mathfrak{p}$. Choose $x \in \mathrm{ann}(W) - \mathfrak{p}$. Then $F_x \cong M_x$, which contradicts the fact that $x \notin I$. Thus, $\mathfrak{m}^N \subset I$, for some $N > 0$. If $y_1, \ldots, y_d$ is a system of parameters, then set $x_i = y_i^N$. □

Remark 4.3. There is a graded version of this result, where “local” can be replaced by graded.

Corollary 4.4. *Let A and M be as above. Then there is a system of parameters $x_1, \ldots, x_d$ and short exact sequences $0 \to M \to F \to T_i \to 0$ where F is A-free and $x_i \cdot T_i = 0$.*

Proof. According to (4.2), choose a system of parameters $y_1, \ldots, y_d$ such that M_{y_i} is A_{y_i}-free. Set $y = y_i$ and write $M_y \cong \bigoplus_{j=1}^r A_y e_j = G$. There is a short exact sequence $0 \to M \to M_y \to W \to 0$, where W is y-torsion. Any generator m_l of M can be expressed as $\sum_{j=1}^r \alpha_{lj} e_j$, for some $\alpha_{lj} \in A_y$. Let t be the maximum power of all the denominators of the coefficients α_{lj}, for all l and j. Set $e'_j = e_j/y^t$. Then $m_l = \sum_{j=1}^r \alpha_{lj} \cdot y^t \cdot e'_j \subset \bigoplus_{j=1}^r Ae'_j$. In other words, M is a subset of a free A-module of rank r. Call this free module F. F/M is y-torsion since $F/M \subset G/M$. Moreover, each generator of F is sent into M by some finite power of y. Let s (i.e., s_i) be the maximum of these powers. Set $x_i = y_i^{s_i}$, $1 \leqslant i \leqslant d$. □

Corollary 4.5. *With the same notation as above, $x_i \cdot \mathrm{Ext}^1_A(M, -) \equiv 0$, for all i.*

Proof. From the short exact sequences of Corollary 4.4, we obtain exact sequences $0 = \mathrm{Ext}^1_A(F, -) \to \mathrm{Ext}^1_A(M, -) \to \mathrm{Ext}^2_A(T_i, -) \to 0$, where $x_i \cdot \mathrm{Ext}^2_A(T_i, -) = 0$. □

Corollary 4.6. *With the same notation as above, if $\mathfrak{a}$ is an ideal such that* $\mathrm{Hom}_A(\mathfrak{a}, M) \cong M$ *then* $\mathbf{x}^{(2)}\, \mathrm{Ext}^1_A(\mathfrak{a}^*, -) \equiv 0$. *(See Section* 3 *for notation.)*

Proof. Set $x = x_i$ and $0 \to M \to F \to T \to 0$, as in (4.4). Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & \mathrm{Hom}_A(\mathfrak{a}, F) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_A(\mathfrak{a}, T). \\ & & & T' \arrow[ur, hook] & \end{tikzcd}$$

The short exact sequence $0 \to M \to \mathrm{Hom}_A(\mathfrak{a}, F) \to T' \to 0$ induces a long exact sequence of functors:

$$\mathrm{Ext}^1_A(T', -) \longrightarrow \mathrm{Ext}^1_A\big(\mathrm{Hom}_A(\mathfrak{a}, F), -\big) \longrightarrow \mathrm{Ext}^1_A(M, -).$$

Observe that $\mathrm{Hom}_A(\mathfrak{a}, F) \cong \bigoplus_{j=1}^r \mathfrak{a}^*$. Since $x_i \cdot T_i' = 0$ and $x_i \cdot \mathrm{Ext}^1_A(M, -) = 0$, we conclude that $x_i^2 \cdot \mathrm{Ext}^1_A(\bigoplus_{j=1}^r \mathfrak{a}^*, -) = 0$. This holds for all i. $\square$

We are now in a position to establish the important conclusion of this section and to apply it to the results from Section 3. Using the above facts, we have the following.

Proposition 4.7. *Let $(A, \mathfrak{m})$ be a normal, local domain of dimension greater than or equal to four such that $\mathrm{Spec}(A) - \mathfrak{m}$ is regular, and let $f_1, f_2, \ldots$ be a sequence of prime elements such that*:

(i) *$f_n \to 0$ in the $\mathfrak{m}$-adic topology, and*
(ii) *each hypersurface $A/f_nA =: A_n$ is normal.*

In addition, suppose A has a small Cohen–Macaulay module M. Then there exists an $N > 0$ such that for all $n \geqslant N$, $\mathrm{Cl}(A) \to \mathrm{Cl}(A_n)$ is injective.

Proof. Because M is locally free on $\mathrm{Spec}(A) - \mathfrak{m}$, we can choose a system of parameters $x_1, \ldots, x_d$ as in (4.2). Let $N > 0$ be such that $f_n \in \mathbf{x}^{(2)}$ for all $n \geqslant N$. If $[\mathfrak{a}] \in \mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}(A_n))$, for $n \geqslant N$, then (3.6) implies that $\mathrm{Hom}_A(\mathfrak{a}, M) \cong M$. By (4.6), for all i, $x_i^2 \cdot \mathrm{Ext}^1_A(\mathfrak{a}^*, -) \equiv 0$. But by (4.1), $f_n \cdot \mathrm{Ext}^1_A(\mathfrak{a}^*, -)$ is NOT identically zero. Therefore, $[\mathfrak{a}^*]$, and hence $[\mathfrak{a}]$, must be trivial, since $f_n \in \mathbf{x}^{(2)}$. $\square$

Next, Theorem 3.5 can be improved in case the family of hypersurfaces is normal. Note that the case of equicharacteristic zero is handled in [21].

Theorem 4.8. *Let k be a perfect field of characteristic $p > 0$. Let S be an $\mathbb{N}$-graded ring of dimension greater than or equal to four such that $S_0 = k$ and S is a normal domain. We assume that $\mathrm{Spec}(S) - \mathfrak{m}$ is regular, where $\mathfrak{m} = S_+$. If $f_1, f_2, \ldots$ is a sequence of homogeneous prime elements such that $f_n \to 0$ in the $\mathfrak{m}$-adic topology and each hypersurface S_n is normal, then there is an integer $N > 0$ such that $\mathrm{Cl}(S) \to \mathrm{Cl}(S_n)$ is injective for $n \geqslant N$.*

Proof. The proof proceeds in a familiar manner. First complete S. Secondly, recall that, by (3.1), there is a small Cohen–Macaulay S-module M. Choose a homogeneous system of parameters $x_1, \ldots, x_d$ that satisfies both (4.2) and (3.2). (Note that powers of the x_i's chosen for (4.2) can always be taken so that the s.o.p. satisfies (3.2) as well.) There is an $N > 0$ such that $f_n \in \mathbf{x}^{(2)}$ for $n \geqslant N$. If $[\mathfrak{a}] \in \mathrm{Ker}(\mathrm{Cl}(S) \to \mathrm{Cl}(S_n))$, then as in (3.6), $\mathrm{Hom}_S(\mathfrak{a}, M) \cong M$. But then $[\mathfrak{a}]$ must be trivial, since $f_n \in \mathbf{x}^{(2)}$. $\square$

5. Examples where $\mathbf{Ker(Cl}(A) \to \mathbf{Cl}((A/fA)'))$ is non-trivial

Most referenced examples for which the kernel of the restriction of divisor classes, $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$, is non-trivial come about in two ways. Either A is a complete intersection of dimension less than or equal to three, or $A = B[\![T]\!]$, as in Danilov's theory [5, Section 1] and [6, Section 5]. To be specific, we are not aware of any examples for which $\dim A \geqslant 4$ and A is a local isolated singularity. In part, this is because the (low-dimensional) examples usually start with A being a hypersurface or complete intersection. If A is a complete intersection, as well as an isolated singularity, with $\dim A \geqslant 4$, then Grothendieck's results [11, Chapter XI] apply, and it follows that A is already a UFD; hence, the injectivity of $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$ is a moot point. To remedy this situation, we appeal to a combination of the Danilov results [5, Section 1] and [6, Section 5] together with those of Spiroff [21, Theorem 3.1].

We begin by letting $(A, \mathfrak{m})$ be any excellent local normal domain which is an isolated singularity and contains a sequence of prime elements $\{\pi_n\}_{n=1}^{\infty}$, such that $\pi_n \to 0$ in the $\mathfrak{m}$-adic topology and each $A/\pi_n A$ is an isolated singularity. In addition, suppose that the group homomorphism $\mathrm{Cl}(A[\![T]\!]) \to \mathrm{Cl}(A)$ is not injective. This last hypothesis is not difficult to achieve, for when A contains a field k of characteristic zero and $\dim A \geqslant 3$, it amounts to requiring that A not satisfy the Serre condition S_3. (See [7, Theorem 2].) That is, we require $\mathrm{H}^1(X, O_X) \neq 0$, where $X = \mathrm{Spec}(A) - \mathfrak{m}$.

Recall from [21] that one has $\mathrm{Cl}(A) \to \mathrm{Cl}((A/\pi_n A)')$ in this case. Moreover, the existence of π_n in A satisfying the requirements above can be obtained in many cases as a result of Bertini's Theorem [9, p. 10]. However, one can make a standard generic construction as we do below within Example 5.2. Finally, we note that the hypotheses on A above do not impose any restriction on the dimension of A, beyond requiring that it be at least three.

Theorem 5.1. *Let $(A, \mathfrak{m}, k)$, for k an algebraically closed field, be an excellent local normal domain which is an isolated singularity and contains a sequence of prime elements $\{\pi_n\}_{n=1}^{\infty}$, such that $\pi_n \to 0$ in the $\mathfrak{m}$-adic topology and each $A/\pi_n A$ is an isolated singularity. In addition, suppose that $\mathrm{Cl}(A[\![T]\!]) \to \mathrm{Cl}(A)$ is not injective. Let $B_n = A[\![T]\!]/(T^n - \pi_n)$, for $n = 1, 2, 3, \ldots$. Then*:

(i) *for n not a multiple of the characteristic of k, the ring B_n is a normal local domain that is an isolated singularity,*
(ii) *there is a natural isomorphism $(B_n/tB_n)' \cong (A/\pi_n A)'$, where t is the image of T in B_n, and*

(iii) *the group homomorphism* $\mathrm{Cl}(B_n) \to \mathrm{Cl}((B_n/tB_n)')$ *is not injective for at least one* $n > 0$.

Proof. (i) Since B_n is the t-adic completion of $A[T]/(T^n - \pi_n)$, it is enough to consider the polynomial version. More specifically, we will show that when $A[T]/(T^n - \pi_n)$ is localized at a prime $\mathfrak{P}$ that contracts to $\mathfrak{p} \in \mathrm{Spec}(A) - \{\mathfrak{m}\}$, then the result is a regular ring. Consequently, by excellence, the completed ring B_n is a local normal ring that is an isolated singularity. Let $\mathfrak{p} \in \mathrm{Spec}(A) - \{\mathfrak{m}\}$.

Consider the case $\pi_n \in \mathfrak{P}$. Then $\pi_n \in \mathfrak{p}$. Note that $A_\mathfrak{p}/\pi_n A_\mathfrak{p}$ is a regular local ring and $A[T]_\mathfrak{P}/(T^n - \pi_n)A[T]_\mathfrak{P}$ is a localization of $A_\mathfrak{p}[T]/(T^n - \pi_n)A_\mathfrak{p}[T]$. From the short exact sequence

$$0 \longrightarrow A_\mathfrak{p}[T]/(T^n - \pi_n)A_\mathfrak{p}[T] \xrightarrow{\cdot t} A_\mathfrak{p}[T]/(T^n - \pi_n)A_\mathfrak{p}[T] \longrightarrow A_\mathfrak{p}/\pi_n A_\mathfrak{p} \longrightarrow 0,$$

one obtains that $A_\mathfrak{p}[T]/(T^n - \pi_n)A_\mathfrak{p}[T]$ is a regular ring. Consequently, the ring $A[T]_\mathfrak{P}/(T^n - \pi_n)A[T]_\mathfrak{P}$ is regular.

Next, suppose $\pi_n \notin \mathfrak{P}$. For n such that $p \nmid n$, π_n is a unit in $A_\mathfrak{p}$ and $T^n - \pi_n$ is a separable polynomial in $\kappa(\mathfrak{p})[T]$. Equivalently [8, p. 114], $T^n - \pi_n$ is separable over $A_\mathfrak{p}$; that is, the extension $A_\mathfrak{p} \to A_\mathfrak{p}[T]/(T^n - \pi_n)A_\mathfrak{p}[T]$ is étale. As a result, $A_\mathfrak{p}[T]/(T^n - \pi_n)A_\mathfrak{p}[T]$ is a regular ring. Therefore, as above, the ring $A[T]_\mathfrak{P}/(T^n - \pi_n)A[T]_\mathfrak{P}$ is regular.

(ii) Observe that $B_n/tB_n \cong A[\![T]\!]/(T^n - \pi_n, T) \cong A/\pi_n A$.

(iii) For each n, there exists a commutative diagram of ring homomorphisms:

$$\begin{tikzcd}
 & B_n \arrow[dr, "\text{mod } t"] & & \\
A[\![T]\!] \arrow[ur, "\text{mod } T^n-\pi_n"] \arrow[dr, "\text{mod } T"'] & & A/\pi_n A \arrow[r, hook] & (A/\pi_n A)' \cong (B_n/tB_n)'. \\
 & A \arrow[ur, "\text{mod } \pi_n"'] & &
\end{tikzcd}$$

As a result, there is a commutative diagram on divisor class groups:

$$\begin{tikzcd}
 & \mathrm{Cl}(B_n) \arrow[dr] & \\
\mathrm{Cl}(A[\![T]\!]) \arrow[ur] \arrow[dr] & & \mathrm{Cl}((B_n/tB_n)'). \\
 & \mathrm{Cl}(A) \arrow[ur] &
\end{tikzcd}$$

Let $[\mathfrak{a}]$ be a non-trivial element of the kernel of $\mathrm{Cl}(A[\![T]\!]) \to \mathrm{Cl}(A)$. Then $[\mathfrak{a}] \notin \bigcap \mathrm{Ker}(\mathrm{Cl}(A[\![T]\!]) \to \mathrm{Cl}(B_n))$, by [21, Theorem 3.1]. Thus, for at least one $n \geqslant 1$, the image of $[\mathfrak{a}]$ in $\mathrm{Cl}(B_n)$ is non-trivial. By commutativity of the diagram, for any n, this image is in the kernel of $\mathrm{Cl}(B_n) \to \mathrm{Cl}((B_n/tB_n)')$. $\square$

Example 5.2. Let S be the graded ring over $\mathbb{C}$ that is the Segre product $\mathbb{C}[X_0, X_1, X_2]/(X_0^l + X_1^l + X_2^l) \times_{\mathrm{Segre}} \mathbb{C}[Y_0, Y_1]$, where l is any integer greater than two. Then $\dim S = 3$ and S is not Cohen–Macaulay due to a theorem of Chow. (See [4, p. 818] and especially [14, Section 14].) By repeating the Segre product with rings of the form $\mathbb{C}[Y_0, Y_1]$, one elevates the dimension by one each time. However, the depth will remain at two as a result of repeated use of the Künneth formula for computing scheme cohomology, [14, Proposition 5.1, Section 14] and [22, Corollary 1.1]. With this process, we can construct $\dim S$ to be as large as we like, where S is now the ring obtained after any certain number of iterations.

The localization of S at its irrelevant graded maximal ideal provides a local normal isolated singularity of depth exactly two. We refer to this ring as A. Let $a_1, \dots, a_d$ be a system of parameters and let $\pi = \sum_{i=1}^d a_i X_i \in A[X_1, \dots, X_d]_{\mathfrak{m}[\underline{X}]}$. It is routine to argue that $A[\underline{X}]_{\mathfrak{m}[\underline{X}]}$ is an isolated singularity. More specifically, set $B = A[X_1, \dots, X_d]$. If any a_i is inverted, then

$$X_i = \frac{1}{a_i}\pi - \sum_{j \neq i} \frac{a_j}{a_i} \cdot X_j \quad \text{and} \quad B_{a_i} = A_{a_i}[X_1, \dots, X_{i-1}, \pi, X_{i+1}, \dots, X_d].$$

Let $\mathfrak{P}$ be a non-maximal prime ideal of $B_{\mathfrak{m}[\underline{X}]}$. Set $\mathfrak{p} = \mathfrak{P} \cap A$. Then some $a_i \notin \mathfrak{p}$. As a result, $B_{\mathfrak{P}} \cong (B_{a_i})_{\mathfrak{P}B_{a_i}}$; i.e., $B_{\mathfrak{P}}$ is a localization of a regular ring. Set $\pi_n = \sum_{i=1}^d a_i^n X_i$. Then the argument that $A[\underline{X}]_{\mathfrak{m}[\underline{X}]}/(\pi_n)$ is an isolated singularity is similar. The important point to note is that $B_{a_i}/\pi_n B_{a_i} \cong A_{a_i}[X_1, \dots, \hat{X}_i, \dots, X_d]$. Further, we see that $\pi_n \to 0$ in the maximal ideal topology on $A[\underline{X}]_{\mathfrak{m}[\underline{X}]}$. We now apply Theorem 5.1.

Appendix A. Unique lifting of small Cohen–Macaulay modules

The discussion here follows that of Yoshino [23, pp. 48–49] and Popescu [18, Theorem 1.2], and is tailored for our needs in the proof of Theorem 3.5. We keep our remarks brief since for the most part we are simply observing that the Cohen–Macaulay hypothesis on the ring can be dropped. For a system of parameters $x_1, x_2, \dots, x_d$, we are using the notation $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{x}^{(2)} = (x_1^2, x_2^2, \dots, x_d^2)$.

Theorem 6.1 (Popescu–Yoshino). *Let $(A, \mathfrak{m})$ be a local ring and suppose the system of parameters $x_1, \dots, x_d$ has the property that $\mathbf{x}\operatorname{Ext}_A^1(M, -) \equiv 0$, whenever M is a small Cohen–Macaulay A-module. If M and N are two small Cohen–Macaulay A-modules, then for any homomorphism $\overline{\phi} : M/\mathbf{x}^{(2)}M \to N/\mathbf{x}^{(2)}N$, there is a homomorphism $\phi : M \to N$ such that $\overline{\phi} \equiv \phi$ (mod $\mathbf{x}$).*

Proof. Since M and N are small Cohen–Macaulay modules, we note that both $\mathbf{x}$ and $\mathbf{x}^{(2)}$ are regular sequences on M and N. Following the notation of Yoshino [23, p. 48], we let $\mathbf{y}_i = (x_1^2, \dots, x_i^2)$ and $\mathbf{z}_i = (x_1^2, \dots, x_i^2, x_{i+1})$. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N/\mathbf{y}_i N \arrow[r, "\cdot x_{i+1}^2"] \arrow[d, "\cdot x_{i+1}"'] & N/\mathbf{y}_i N \arrow[r] \arrow[d, equal] & N/\mathbf{y}_{i+1} N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N/\mathbf{y}_i N \arrow[r, "\cdot x_{i+1}"] & N/\mathbf{y}_i N \arrow[r] & N/\mathbf{z}_i N \arrow[r] & 0
\end{tikzcd}$$

and the functor $\mathrm{Hom}_A(M, -)$ yield the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(M, N/\mathbf{y}_i N) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(M, N/\mathbf{y}_{i+1} N) \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(M, N/\mathbf{y}_i N) \arrow[d, "\cdot x_{i+1}"] \\
\mathrm{Hom}_A(M, N/\mathbf{y}_i N) \arrow[r] & \mathrm{Hom}_A(M, N/\mathbf{z}_i N) \arrow[r] & \mathrm{Ext}^1_A(M, N/\mathbf{y}_i N).
\end{tikzcd}$$

Since the multiplication map $\cdot x_{i+1}$ in the far right column represents the zero map, one gets that, for any $\phi_{i+1} \in \mathrm{Hom}_A(M, N/\mathbf{y}_{i+1}N)$, there is $\phi_i \in \mathrm{Hom}_A(M, N/\mathbf{y}_i N)$ such that ϕ_i agrees with ϕ_{i+1} modulo $\mathbf{z}_i$. Since for all i, $\mathrm{Hom}_A(M, N/\mathbf{y}_i N) = \mathrm{Hom}_A(M/\mathbf{y}_i M, N/\mathbf{y}_i N)$, the claim follows by induction, by successively lifting from $\phi_{i+1} : M/\mathbf{y}_{i+1}M \to N/\mathbf{y}_{i+1}N$ to $\phi_i : M/\mathbf{y}_i M \to N/\mathbf{y}_i N$. □

Remark 6.2. The above result holds when "local" is replaced by "graded"; that is, when all rings and modules are graded, $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$, with A_0 a field, and the maximal ideal $\mathfrak{m}$ is the irrelevant one, namely A_+.

Corollary 6.3. *Notation is the same as in* (6.1). *If $f \in \mathbf{x}^{(2)}$, then a homomorphism $\overline{\psi} : M/fM \to N/fN$ "lifts" to a homomorphism $\phi : M \to N$ such that ϕ and $\overline{\psi}$ agree modulo* $\mathbf{x}$.

Proof. Note that $\overline{\psi}$ induces a homomorphism $\bar{\phi} : M/\mathbf{x}^{(2)}M \to N/\mathbf{x}^{(2)}N$. From (6.1), $\bar{\phi}$ is induced by a homomorphism $\phi : M \to N$, where $\bar{\phi} \equiv \phi$ (mod $\mathbf{x}$). Since $\overline{\psi}$ and $\bar{\phi}$ agree modulo $\mathbf{x}^{(2)}$, it follows that $\overline{\psi}$ and ϕ agree modulo $\mathbf{x}$. □

Corollary 6.4. *Notation is the same as in* (6.1). *If there is a prime element $f \in \mathbf{x}^{(2)}$ such that $M/fM \cong N/fN$, then $M \cong N$; i.e., lifting small Cohen–Macaulay modules modulo f is unique* (*when it occurs*).

Proof. From (6.3), an isomorphism $\overline{\psi} : M/fM \to N/fN$ lifts to a homomorphism $\phi : M \to N$ such that $\overline{\psi} \equiv \phi$ (mod $\mathbf{x}$). More specifically, for any $n \in N$, there exists an $m \in M$ such that $\phi(m) + \mathbf{x}N = n + \mathbf{x}N$; i.e., $N = \phi(M) + \mathbf{x}N$. Thus, by Nakayama's Lemma, ϕ is surjective. Since the sequence $x_1, \ldots, x_d$ is N-regular, applying $\otimes_A A/\mathbf{x}$ to the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\phi) \longrightarrow M \xrightarrow{\phi} N \longrightarrow 0,$$

we obtain another short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\phi) \otimes_A A/\mathbf{x} \longrightarrow M \otimes_A A/\mathbf{x} \xrightarrow{\phi \otimes A/\mathbf{x}} N \otimes_A A/\mathbf{x} \longrightarrow 0.$$

Since $\overline{\psi} \equiv \phi$ (mod $\mathbf{x}$) and $\overline{\psi}$ is an isomorphism, $\mathrm{Ker}(\phi) \otimes A/\mathbf{x} = 0$. Thus, by Nakayama's Lemma, $\mathrm{Ker}(\phi) = 0$. □

Once again, we remark that (6.4) holds in the graded setting of (6.2). One may develop much more. Consult [23, Section 6] and [18, Section 1].

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